PART ONE

THE ANALYTIC HIERARCHY PROCESS

THEME Decomposition by hierarchies and synthesis by finding relations through informed judgment.

When economic factors have been reduced to numbers measured in dollars, when numbers of objects, their weight in tons, and the time needed to produce them have been calculated and when probabilities have been estimated, our modeling of complex human problems often would have reached the limits of its effectiveness. It depends strongly on what factors we can measure.

If then the models do not work well because we have left out significant factors by making simplifying assumptions, at least in the social sciences, we blame the result on politics and on capricious human behavior and other factors regarded as annoying aberrations of human nature which will disappear in time. But these are precisely the controlling factors that we must deal with and measure in order to get realistic answers. We must stop making simplifying assumptions to suit our quantitative models and deal with complex situations as they are. To be realistic our models must *include* and *measure* all-important tangible and intangible, quantitatively measurable, and qualitative factors. This is precisely what we do with the analytic hierarchy process (AHP). We also allow for differences in opinion and for conflicts as the case is in the real world. We intend to develop this approach and show the reader how effective it is as a tool.

Chapter 1 gives a general introduction to the subject followed by examples in Chapter 2. Chapter 3 provides background on scales and consistency and Chapter 4 is concerned with hierarchic structures and their consistency.

CHAPTER

ONE

HIERARCHIES AND PRIORITIES: A FIRST LOOK

1-1 INTRODUCTION

When we think, we identify objects or ideas and also relations among them. When we identify anything, we decompose the complexity which we encounter. When we discover relations, we synthesize. This is the fundamental process underlying perception: decomposition and synthesis. The elaboration of this concept and its practical implications interest us here.

We all experience reality sufficiently close, so that though our decompositions of it may be differ, our evaluations at the operational level tend to be close, particularly when it is supported by successful experience in fulfilling our common purposes. Thus we may model reality somewhat differently, but we manage to communicate a sense of judgment, which involves common understanding (but not without differences). We need to exploit this manifestation of judgment and of learning.

Our purpose is to develop a theory and provide a methodology for modeling unstructured problems in the economic, social, and management sciences. Sometimes we forget how long it took the human race to evolve measurement scales that are useful in daily living. The evolution of monetary units with wide acceptance has taken thousands of years of barter and legislation in a successive approximation process to design a medium for exchange, which we call money. Money also serves as a basis of measurement of all kinds of goods and services. This evolution of a measurement scale, i.e., money, has helped to structure economic theory, making it amenable to empirical tests. The development of money has been an interactive process refining human judgment and experience on the one hand, and the medium of measurement on the other. This process has also established a framework, which incorporates both philosophy and mathematics in the challenging science of economics. Economic theory is tied today very strongly to its unit of measurement, but has problems in dealing with political and social values that do not have economic implications.

Social values in our complex society call for a convenient method of scaling to enable us, on a daily basis, to evaluate tradeoffs between money, environmental quality, health, happiness, and similar entities. Such an approach should facilitate interaction between judgment and the social phenomena to which it is applied. We need such an approach precisely because they are no social measurement scales that have acquired popular use, although for a theory of measurement.

The acid test for a new tool is how “natural” and easy it is to understand and how well it *integrates* within existing theory, whether it is accepted by those who need to use it, and how well it *works* in solving their problems.

Our theory was developed to solve a specific problem in contingency planning (Saaty, 1972) and a later major application was to alternative futures for a developing country, the Sudan (Saaty, 1977d). The result was a set of priorities and an investment plan for projects to be undertaken there in the late 1980s. The ideas have gradually evolved through use in a number of other applications ranging from energy allocation (Saaty and Mariano, 1979), investment in technologies under uncertainty, dealing with terrorism (Saaty and Bennett, 1977), buying a car, to choosing a job and selecting a school. Using pairwise comparison judgments for input, we can cope (in what we see as a natural way) with factors which, in the mainstream of applications, have not been effectively quantified. Naturally, one has to be concerned with the ambiguity, which occurs whenever numbers are associated with judgment; otherwise, one may fall in the trap of the modern epithet: "Garbage in, garbage out." Judgment is difficult to work with and widely variable. But we can study the consistency of judgment and its validity.

Various applications of the theory have involved the participation of lawyers, engineers, political, social and physical scientists and mathematicians, and even children. All felt comfortable with the easy and natural way they were to provide pairwise comparisons in their area of expertise, and with the explanation of the method which was usually interpreted to them non-technically.

But why this obsession with numbers and measurement? How do we hope it will help us and how will it work? We constantly offered techniques to cope with every phenomenon we face today. But the techniques cannot cope with entities for which there are not measures. Here we have an effective way to create measures for such entities and then use them in decision-making.

Our approach, which is sufficiently general to use both known measurement and judgment, may be better appreciated through the following quotation (Churchman and Eisenber, 1969):

> …It seems almost obvious that we cannot solve present-day major political and organizational problems simply by grinding through a mathematical model or computer a set of accepted inputs. What we require besides is the design of better deliberation and judgment. Once we begin to understand the process of deliberation and judgment we may on a better objective method, that is, a way to express optimal deliberation in a precise and warranted form.

In general, decision making involves the following kinds concerns: (1) planning; (2) generating a set of alternative; (3) setting priorities; (4) choosing a best policy after finding a set of alternatives; (5) allocating resources; (6) determining requirements; (7) predicting outcomes; (8) designing systems; (9) measuring performance; (10) insuring the stability of a system; (11) optimizing; and (12) resolving conflict.

Solving decision problems has suffered from an overabundance of "patent medicine" techniques without any holistic cure. The recommendations to solve one problem may leave the whole system more disturbed than it was to begin with.

In recent decades, the "system approach" to problems in the social and behavioral sciences has found its place next to the older reductionist methods, which seem more appropriate to the physical sciences. Basically, a system is an abstract model for a real-life structure such as nervous system of a human, the government of a city, the transportation network of a state, or the ecosystem of the marshlands of New Jersey. In systems language we evaluate the impact of various components of a system on the entire system and find their priorities.

Some people have defined a system in terms of the interactions of its parts. But a much richer definition of a system in terms of its *structure*, its *functions*, the objectives set for it in the *design* from the *perspective* of a particular individual or group (hence the possibility for conflict), and finally the *environment* (the larger surrounding system) of which it is a subsystem. For practical purposes a system is often in terms of its

(1) *Structure* according to the physical, biological, social, or even psychological arrangement of its parts and according to the flow of material and people which define the relations and dynamics of the structure, and
(2) *Function* according to what functions the components of the system, whether animate or inanimate, are meant to serve; what these functions are and what objectives they are intended to fulfill; what higher objectives these objectives are part of (leading up to an overall purpose of the system); whose objectives are being satisfied; what conflict among individuals may have to be resolved.

Actually, the structure and function of a system cannot be separated. They are the reality we experience. What we would like to do is look at them simultaneously. In doing this, the structure serves as a vehicle for analyzing the function. The functioning modifies the dynamics of the structure.

A hierarchy is an abstraction of the structure of a system to study the functional interactions of its components and their impacts on the entire system. This abstraction can take several related forms, all of which essentially descend from an apex (an overall objective), down to sub-objective, down further to forces which affect these sub-objectives, down to the people who influence these forces, down to objectives of the people and then to their policies, still further down to the strategies, and finally, the outcomes which result from these strategies. It is worth noting that there is a degree of invariance to this structure whose highest levels represent the environmental constraints and forces descending to levels of actors, their objectives, the functions of the system, and, finally, to its structure which may be modified or controlled.

Two questions arise in the hierarchical structuring of systems:

(1) How do we structure the functions of a system hierarchically?
(2) How do we measure the impacts of any element in the hierarchy?

There are also relevant questions of optimization with which we may wish to deal. They are meaningful after we have answered the above questions. We shall have a number of things to say later about the structure of hierarchies.

1-2 MEASUREMENT AND THE JUDGMENTAL PROCESS

Let us examine three related problems which have interesting applications. The *first* is concerned with measurement. Suppose we are given a set of objects which are all sufficiently light and can be lifted by hand. In the absence of a weighting instrument we wish to estimate their relative weights. One way could be to guess the weight of each object directly in pounds, for example, by lifting it (perhaps using the lightest one as standard), comparing the whole class, and then dividing the weight of each by the total to gets its relative weight. Another method which utilizes more of the available information in the experiment is to compare the objects in pairs, by lifting one and then lifting another and then back to the first and then again the second and so on until we have formulated a judgment as to the relative weight (ratio) of each pair of objects. The problem then is to adopt a meaningful scale for the pair comparisons. This second process has the advantage of focusing exclusively on two objects at a time on how they relate to each other. It also generates more information than is really necessary since each object is methodically compared with every other.

For problems where there is no scale to validate the result, the pairwise comparison process can prove to be an asset, because although the steps are more numerous, they are simpler than in the first process.

We note that consistency in any kind of measurement cannot be taken for granted. All measurement, including that which makes use of instruments, is subject to experimental error and to error in the measuring instrument. A serious effect of error is that it can and often does lead to inconsistent conclusions. A simple example of the consequence of error in weighting object is to find that *A* is heavier than *B*, and *B* is heavier than *C* but *C* is heavier than *A*. This can happen particularly when the weights of *A, B* and *C* are close, and the instrument is not fine enough to distinguish them. Lack of consistency may be serious for some problems but not for others. For example, if the objects are two chemicals to be mixed together in exact proportion to make a drug, inconsistency may mean that proportionately more of one chemical is used than the other, possibly leading to harmful results in using the drug.

But perfect consistency in measurement, even with the fines instruments, is difficult to attain in practice; what we need is a way of evaluating how bad it is for a particular problem.

By consistency we mean here not merely the traditional requirement of the transitivity of preferences (if apples are preferred to oranges and oranges are preferred to bananas, then apples must be preferred to bananas), but the actual intensity with which the preference is expressed transits through the sequence of objects in comparison. For example, if apples are twice as preferable as oranges and oranges are three times as preferable as bananas, then apples must be six times as preferable as bananas. This is what we call cardinal consistency in the strength of preference. Inconsistency is a violation of proportionality which may or may not entail violation of transitivity. Our study of consistency demonstrates that it is not whether we are inconsistent on particular comparisons that matter, but how strongly consistency is violated in the numerical sense for the overall problem under study. An exact definition f a numerical index for consistency will be given later.

Note that there need be no relationship between consistency and tests of how closely a measurement duplicates reality. Thus, an individual may have excellent consistency but not know anything about the real situation. Usually, though, the more a person knows a situation, the more consistent one would expect him to be in representing it. Pairwise comparisons enable one to improve consistency by using as much information as possible. To represent reality with measurements, we assume the following:

(1) At least physical "reality" is consistent and can be counted on to yield similar results from trial under controlled conditions.
(2) Judgment must strive towards consistency. Consistency is a desirable objective. It is necessary for capturing reality but not sufficient. An individual may have very consistent ideas which do not correspond to the "real" world situation. Consistency is a central question in concrete measurement, in judgments, and in the thinking process.
(3) To obtain better estimates of reality, we should channel our impressions. Feelings, beliefs in a systematic way in providing judgments. The object is to enhance objectivity and downplay too much subjectivity.
(4) To get good results (which correspond to reality) from our feelings we need: (*a*) to use mathematics to construct the right kind of theory to produce numerical scales of judgments and other comparative measurements, (*b*) to find a scale which discriminates between our feelings, whose values have some kind of regularity so that we can easily rely on making the correspondence between our qualitative judgments and these numbers, (*c*) to be able to reproduce the measurement of reality which we have already learned in physics and economic, and (*d*) to be able to determine how inconsistent we are.

In passing, we note that measuring instruments are not and cannot be means of absolute measurement, but they have been the object of scientific study and analysis and have been constructed with consistent behavior in mind, and have come to serve as vehicles in other scientific research. If these instruments are for any reason inadequate (and one can always devise an experiment for which there is no satisfactory instrument for measurement) then we must keep inventing new instruments. It is not difficult to imagine some important experiment for which no sufficiently fine instrument can ever be found from which consistent answers can always be obtained. In that case entire problem is shifted to the study of consistency and evaluating the seriousness of inconsistency. The maximum eigenvalue approach to estimate ratio scales which we study here gives rise to a measure of departure from consistency enabling comparison between informed and random or unrelated judgments and serves as a vehicle for estimating departure from the underling ratio scale.

In the measurement of physical quantities it is usually possible to set down a dimension or property such as length, which remains the same in time and space, and devise instruments to measure this property. Naturally it would be more difficult to make an instrument which adjusts its scale to changing circumstances before making a measurement. For example, length and mass vary at speeds near that of light and an instrument that directly measured these properties at near the speed of light may require some kind of variable scale.

This is precisely the problem in the social science. When we deal with properties that change not only in time and space but also (and far more seriously) in conjunction with other properties, their meaning also changes. We cannot improvise universal scales for social events. Social phenomena are more complicated than physical phenomena because they are harder to replicate in abundance. Too much control must be imposed and controls in themselves often destroy the very social behavior one is trying to measure. Our judgments must be sufficiently flexible to take into consideration the contextual setting of the property being measured.

Consider the problems of measuring achievement and happiness. Both may be called relative properties in that the unit of measurement may have to be adjusted to compare, for example, the degree of happiness in one setting with that in another. As we shall see, it is possible to do this with the pairwise comparison technique. A powerful instrument which varies its scales with the relativeness of the circumstance can be the human mind itself, particularly if it turns out that its measurement is sufficiently consistent to satisfy the requirements of the particular problem. The intensity of our feelings serves as a scale-adjustment device to put the measurement of some objects on a scale commensurate with that of other objects. In fact, as the mind improves its precision, it becomes the required tool for relative measurement as no instrument except our very personal designed one (our own mind) can be made to suit our particular experience and viewpoint. A group must coordinate their outlook to produce results acceptable (in some sense) to them.

We now return to our *second* problem, which is concerned with providing greater stability and invariance to social measurement. Granted that dimensions or properties are variable, how do we measure the impact of this variability on still other higher level properties, and, in turn, these in still higher ones. It turns out that for a very wide class of problems we can usually identify overall properties (or one property), which remain the same sufficiently long, i.e., for the duration of an experiment. This approach leads us to the measurement and analysis of impacts in hierarchies as discussed earlier.

We can then study the invariance of the derive measures by reorganizing the hierarchy in different ways. The results of the measurement may be used to stabilize the system or to design new goal-oriented systems. They can also be used (as priorities) to allocate resources.

Here again, as in the monetary system described earlier, the measurements derive from judgments based on experience and understanding. These measurements are obtainable only from relative comparison and not in an absolute way.

Our *third* problem is to set up the right conditions for people to structure their problems and to provide the necessary judgment to determine their priorities.

We assume that the pairwise comparisons are obtained by direct questioning of people (a single individual if the problem is his sole concern) who may or may not be experts, but who are familiar with the problem. A central point in our approach is that people are often inconsistent, but priorities must be assigned and things done despite inconsistency.

We also assume that all the alternatives are specified in advance, and that not all the variables need to be under the control of each of the parties involve in affecting the outcomes of the alternatives. It is desirable to know if the priority of an alternative is due to the influence of a more powerful outside party. The object may be to improvise policies and establish communication to influence that party to produce a more favorable outcome to the stakeholders. The stability of the results due to changes in judgment evaluation is of interest.

The expressed preferences are assumed to be deterministic rather than probabilistic. Thus, a preference remains fixed and is not contingent upon other factors not included in the problem.

If several people are involved, they can assist each other in sharpening their judgments and also divide the task to provide the judgments in their areas of expertise, thus complementing each other. They may attempt consensus. Failing that, a bargaining process, particularly for people in a dispute, enables one group to yield when the pair being compared is of no significance to them and in return ask for similar concessions from the opposition when that party's interest is involved. When of several individuals does his own evaluations, the separate results may be compared from the standpoints of their individual utilities to obtain a synthesis performed by an outside party of what they would do jointly.

Still another way to use the method would be to have each member of a group with conflicting interests develop the outcome using his judgments and assuming judgments for the other parties, note the outcome, and compare it (perhaps with the aid of a computer) with what the others arrive at. The process reveals what outcome each party is exerting pressure to achieve. The crucial upshot of this is to induce cooperation.

1-3 HIERARCHIES

Very often, as one analyzes the structure if interest, the number of entities and their mutual relations increases beyond the ability of the researcher to comprehend distinct pieces of information. In such cases, the larger the system is broken up into subsystems, almost as the schematic of a computer consists of blocks and their interconnections, with each block having a schematic of its own.

Figure 1-1 represents a very rough representation of the various subsystems which, in their collection and interrelations, make up the trade system of a country today. We will be treating systems like this (which have cycles) in a later chapter.

Now let us turn to the more straightforward hierarchical representation of problems.

A hierarchy is a particular type of system, which is based on the assumption that the entities, which we have identified, can be grouped into disjoint sets, with the entities of one group influencing the entities of only one other group (in a separate chapter we study the interaction between several groups), and being influenced by the entities of only one other group. The elements in each group (also called level, cluster, stratum) of the hierarchy are assumed to be independent. If there is dependence among them we study independence and dependence separately and combine the two as in Chap. 6. The following is an elementary example of a hierarchy.

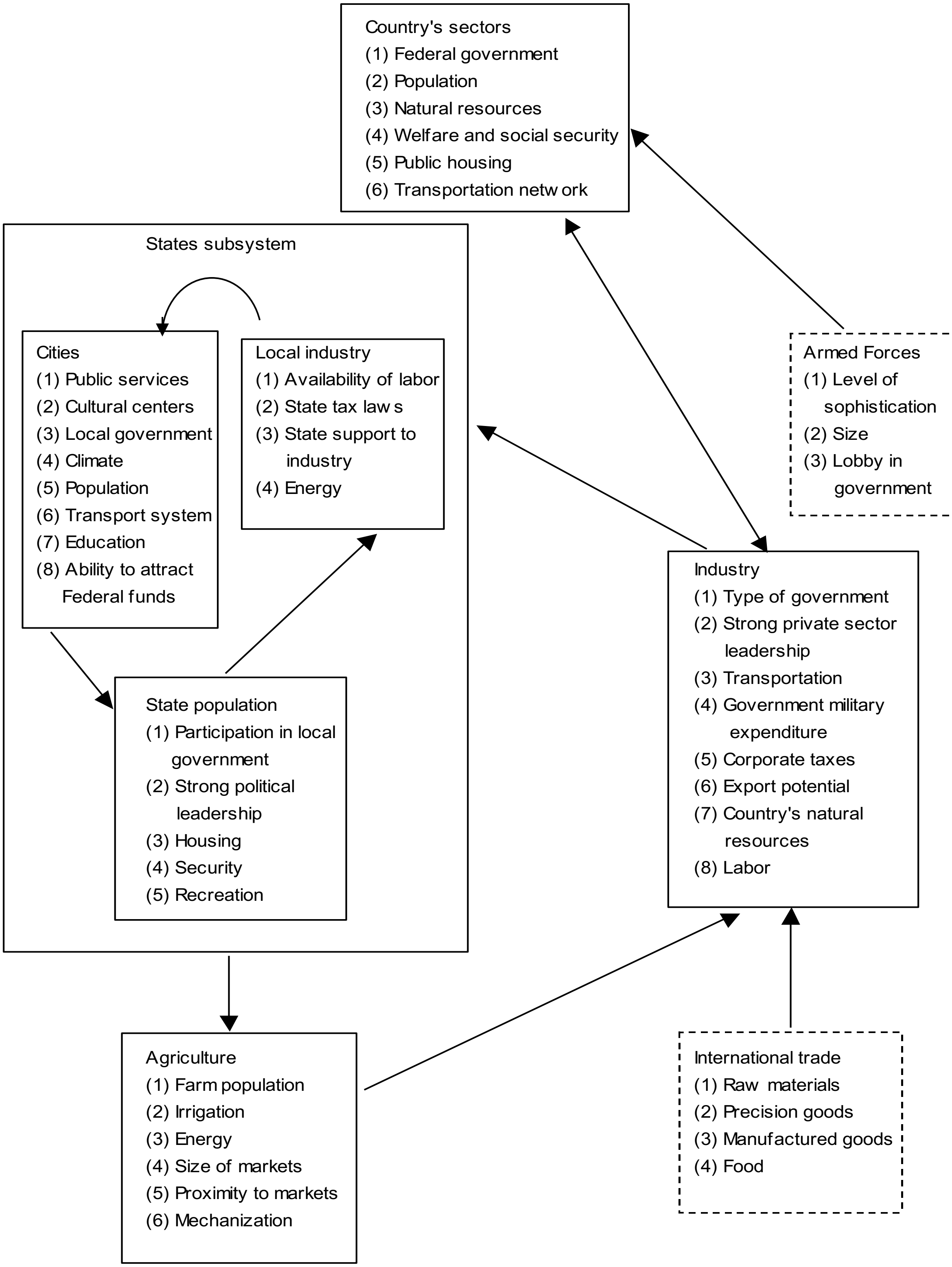
Country's sectors
(1) Federal government
(2) Population
(3) Natural resources
(4) Welfare and social security
(5) Public housing
(6) Transportation netw ork
States subsystem
Cities
(1) Public services
(2) Cultural centers
(3) Local government
(4) Climate
(5) Population
(6) Transport system
(7) Education
(8) Ability to attract Federal funds
Local industry
(1) Availability of labor
(2) State tax law s
(3) State support to industry
(4) Energy
State population
(1) Participation in local government
(2) Strong political leadership
(3) Housing
(4) Security
(5) Recreation
Armed Forces
(1) Level of sophistication
(2) Size
(3) Lobby in government
Industry
(1) Type of government
(2) Strong private sector leadership
(3) Transportation
(4) Government military expenditure
(5) Corporate taxes
(6) Export potential
(7) Country's natural resources
(8) Labor
Agriculture
(1) Farm population
(2) Irrigation
(3) Energy
(4) Size of markets
(5) Proximity to markets
(6) Mechanization
International trade
(1) Raw materials
(2) Precision goods
(3) Manufactured goods
(4) Food

Figure 1-1

The welfare of the city-states of medieval Europe depended mostly on the strength and ability of their rules. The general structure of a city-state may be represented in the hierarchical form shown in Fig. 1-2.

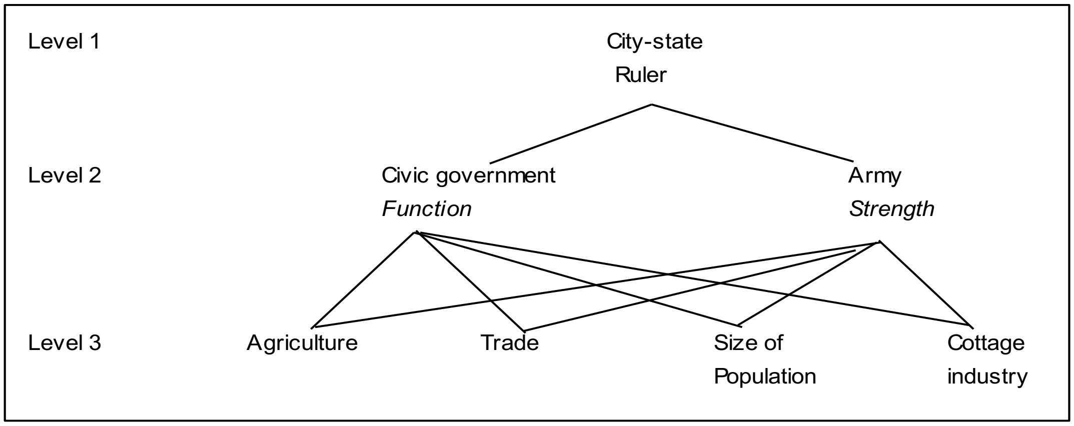

Figure 1-2

We have grouped agriculture, trade, size of population, and cottage industry into one set, or level, because in this model they share the property of being the most fundamental factors in the economic strength of the city-state. They determine the strength of the civic government function, and the army; these two, in turn, influence the welfare of the city-state.

Several observations are in order. Obviously, the model is too simplistic; many more entities could be identified, and more levels. This we can do depending on what question we are attempting to answer. The model can easily expand in complexity and become tedious to deal with. Thus we should construct the hierarchy carefully, choosing between faithfulness to reality and our understanding of the situation from which we can obtain answers. Experience has shown that even seemingly too rough an idealization can yield significant insights.

Second, we have not incorporated the evident fact that not only is the civic government influenced by trade, for instance, but civic government also has its impact on trade. This “reverse” impact, or feedback, while often important, is not as significant as it may seem at first. We have analyzed several problems first without taking feedback into account, and then with feedback. The first results were sufficiently close to allow the assumption that a well-constructed hierarchy will, in most cases, be a good model of reality even if possible feedback relations are ignored. However, as the initial example of this section indicates, some situations may be so complex that their representation by a hierarchy may be deceptively simplistic.

Perhaps another example will further clarify the notion of a hierarchy. The reality we are interested in is a college; wee seek to determine the scenario which will most likely secure the continued existence of the college. Let us call the Focus the welfare of the college. It is influenced by the following forces: instruction, social life, spirit, physical plant, and extracurricular activities. These forces are determined by the following actors: academic administration, non-academic administration, faculty, students, and trustees. We omit the obvious feedback between forces and actors. The various actors have certain objectives; for instance, the faculty may want to keep their jobs, grow professionally, offer good instruction, or, the students may be interested in obtaining a job, getting married, becoming educated, etc. Finally, there are various possible scenarios, such as: status quo, emphasizing vocational training, or continuing education, or becoming a bible school. The scenarios determine the likelihood of achieving objectives, the objectives influence the actors, and the actors guide the forces, which, finally, impact on the welfare of the college. Thus we have the hierarchy of Fig. 1-3.

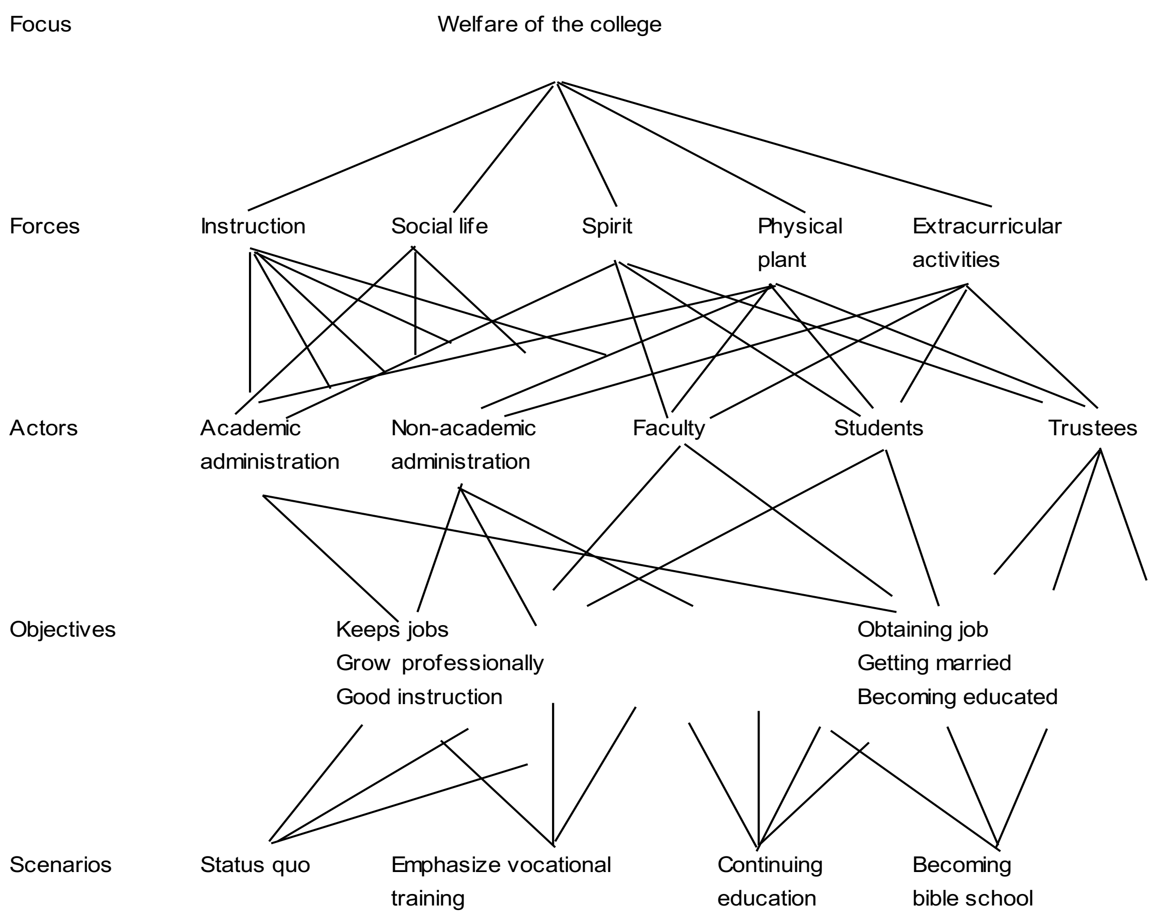

Figure 1-3

Let us give this concept of hierarchy a closer look.

We have a tendency to think that hierarchies were invented in corporations and governments to take care of their affairs. This is not so. These hierarchies are basic to the human way of breaking reality into clusters and sub clusters. Here is a brief eloquent expression in defense of this point of view.

“The immense scope of hierarchical classification is clear. It is the most powerful method of classification used by the human brain-mind in ordering experience, observations, entities and formation. Though not yet definitely established as such by neurophysiology and psychology, hierarchical classification probably represents the prime mode of coordination or organization (i) of cortical processes, (ii) of their mental correlates, and (iii) of the expression of these in symbolisms and languages. The use of hierarchical ordering must be as old as human thought, conscious and unconscious…" (Whyte, 1969).

The basic problem with a hierarchy is to seek understanding at the highest levels from interactions of the various levels of the hierarchy rather than directly from the elements of the levels. Rigorous methods for structuring systems into hierarchies are gradually emerging in the natural and social sciences and in particular, in general systems theory as it relates to the planning and design of social systems.

Direct confrontation of the large and the small is avoided in nature through the use of a hierarchical linkage (see Simon, 1962; Whyte et al., 1969). Conceptually, the simplest hierarchy is linear, rising from one level to an adjacent level. For example, in a manufacturing operation there is a level of workers dominated by a level of supervisors, dominated by a level of managers, on to vice presidents and presidents. A nonlinear hierarchy would be one with circular arrangements so that an upper level might be dominated by a level as well as being in a dominant position (e.g., in case of flow of information). In the mathematical theory of hierarchies we develop a method for evaluating the impact of a level on an adjacent upper level from the composition of the relative contributions (priorities) of the elements in that level with respect to each element of the adjacent level. This composition can be extended upwards through the hierarchy.

Each elements of a hierarchy may belong functionally to several other different hierarchies. A spoon may be arranged with other spoons of different sizes in one hierarchy or with knives and forks in a second hierarchy. For example, it may be a controlling component in a level of one hierarchy or it may be simply unfolding of higher or lower order functions in another hierarchy.

Advantages of Hierarchies

1) Hierarchical representation of a system can be used to describe how changes in priority at upper levels affect the priority of elements in lower levels.
2) They give great detail of information on the structure and function of a system in the lower levels and provide an overview of the actors and their purposes in the upper levels. Constraints on the elements on a level are best represented in the next higher level to ensure that they are satisfied. For example, nature may be regarded as an actor whose objectives are the use of certain material and subject at certain laws as constraints.
3) Natural systems assembled hierarchically, i.e. through modular construction and final assembly of modules; evolve much more efficiently than those assembled as a whole.
4) They are stable and flexible; stable in that small changes have small effect and flexible in that additions to a well-structured hierarchy do not disrupt the performance.

How to Structure a Hierarchy

In practice there is a no set procedure for generating the objectives, criteria, and activities to be included in a hierarchy or even a more general system. It is a matter of what objectives we choose to decompose the complexity of that system.

One usually studies the literature for enrichment of ideas, and often, by working with others, goes through a freewheeling brainstorming session to list all concepts relevant to the problem without regard to relation or order. One attempts to keep in mind that the ultimate goals need to be identified at the top of the hierarchy; their sub-objectives immediately below; the forces constraining the actors still below that. This dominates a level of the actors themselves, which in turn dominates a level of their objectives; below which is a level of their policies, and at the bottom is a level of the various possible outcomes (scenarios). (Refer to the college hierarchy, in Fig. 1-3). This is the natural form that planning and conflict hierarchies take. When designing a physical system, the policies can be replaced by methods of construction. This needs to be followed by several intermediate levels culminating in alternative systems. Considerable criticism and revision may be required before a well-defined plan is formulated.

There is sufficient similarity between problems that one is not always faced with a completely new task in structuring a hierarchy. In a sense the task for the experienced becomes one of identifying the different classes of problems which arise in life systems. There are such a variety of these that the challenge is to become versed with the ideas and concepts which people, living within such a system, encounter. This requires intelligence, patience, and the ability to interact with others to benefit from their understanding and experience.

The overall purpose and other criteria if a hierarchy in socio-political applications may not be unique. They depend on what to examine. This situation is not peculiar to hierarchies and is intrinsic to life situations. For example, in chess we have what is known as the *constant* (a priori) values of the pieces useful in the opening game. There are also the *current* (a posteriori) or empirical values of the pieces as they engage in the confrontations of the end game. Both types of values may be obtained under the following two assumptions (1) in terms of how many squares they control when places on each square and (2) in terms of their being able to check the king without being captured. We have the following relative values for the knight, bishop, rock, and queen (ball, 1947, p. 162).

	Case 1 Controlling squares	Case 2 Threatening king
Constant value	3, 5, 8, 13	12, 13, 24, 37
Current value (empirically)	350, 360, 540, 1000	12, 13, 18, 33

Although the results in Case 2 are close, those in case 1 are different. The analysis gives rise to the question: “What really is the relative worth of pieces in chess?” Obviously there is not a unique answer. However, in terms of relative orders of magnitude the answer may be acceptable.

Our sensory perception operates in specialized ways to serve our survival needs. Therefore, however we try to be objective in interpreting experience, our understanding is perceived and abstracted in a very subjective way, normally to serve our needs! Our survival seems to be a meaningful basis for devising purposes. Shared subjectivity in interpretation is actually what we mean by objectivity. Thus the hierarchies we form are objective by our own definition because they relate to our collective experience.

A valuable observation about the hierarchical approach to problem solving is that the functional representation of a system may differ from person to person, but people usually agree on the bottom level of alternative actions to be taken and the level above it, the characteristics of these actions. For example, the bottom level may consist of alternate traffic routes which can be taken between two points, and the level of characteristics may be include travel time, bottlenecks, potholes, safety, and the like. Table 1-1 indicates levels for different types of hierarchy must always be comfortable that the levels relate naturally to each other. If necessary a level may be expanded into two levels or more completely taken out.

Table 1-1 The general format for hierarchies and decomposition

Generic for a system	Environmental constraints and forces	Perspective (actors)	Objectives of actors	Policies	Outcomes	Resultant outcome
Hierarchy for Conflict	Constraints	Actors	Objective	Policies	Outcomes	Compromise or stable outcome
Forward or projected planning	Present organizational policies →	Other actors →	Other actors objectives →	Policies →	Scenarios →	Logical future ↓
Backward or idealized planning	Organizational response policies ↑ ←	Other actors ←	Other actor objectives ←	Other actor policies ←	Scenarios ←	Desired future
Portfolio Cost-benefit Analysys	Criteria	Sub-criteris	Objectives	Policies	Options	Best option or mix
Investment choice	Risk level	Major forces	Criteria	Problem areas	Specific projects	
Prediction	Risk level	Major forces	Criteria	Problem areas	Categories	

1-4 PRIORITY IN HIERARCHIES

A hierarchy, as presented in the last section, is a more or less faithful model of a real-life situation. It represents our analysis of the most important elements in the situation and of their relationships. It is not a very powerful aid in our decision-making or planning process. What we yet need is a method to determine the potency with which the various elements in one level influence the elements on the next higher level, so that we may compute the relative strength of the impacts of the elements of the lowest level on the overall objectives.

By way of clarification, let us return to the college hierarchy of the last section. As stated here, we are interested in "the scenario which will most likely secure the continued existence" of college. In order to determine this scenario, we begin by finding the strength of importance of the forces with respect to the focus. Then, for each force in turn, we determine the strength of influence of the actors on that force. Simple computation will give us the strength of influence of the actors on the focus. Then, we find the strength of the objectives for each actor; influence of the actors on the focus. Then, we find the strength of the objectives for each actor; and finally, we determine, with respect to each objective, the efficacy of the various scenarios in assuring that objective. Repetition of the computation mentioned above several times will yield the "best" scenarios.

How, then, do we determine the "strengths", or the priorities, of the elements in one level relative to their importance for an element in the next level. At this point, we will present only the most elementary aspects of our method. The psychological motivation for our approach and the mathematical foundation will have to wait.

A few terms must be introduced first. A matrix is an array of numbers, arranged in a rectangle, as in

$$\begin{bmatrix} 1 & 0 & 2.9 & 6 \\ 3 & 3.5 & 7 & 1 \\ 2.1 & 2 & 0 & 1.1 \end{bmatrix}$$

A horizontal sequence of numbers in a matrix is called a row, a vertical one is called a column. A matrix consisting of one row or one column only is called a vector. A matrix is called a square matrix if it has an equal number of rows and columns. It is useful to note that associated with a square matrix are its eigenvectors and corresponding eigenvalues. The reader need not be discouraged with these concepts as we will be developing and explaining them at length in other chapters.

Our method can now be described as follows. Given the elements of one level, say, the fourth, of a hierarchy and one element, *e*, of the next higher level, compare the elements of level 4 *pairwise* in their strength of influence on *e*. Insert the agreed upon numbers, reflecting the comparison, in a matrix and find the eigenvector with the largest eigenvalue. The eigenvector provides the priority ordering, and the eigenvalue is a measure of the consistency of the judgment.

Let us determine a priority scale in the following example. Let *A, B, C, D* stand for chairs, arranged in a straight line, leading away from a light. We develop a priority scale of relative brightness for the chairs. Judgments will be obtained from an individual who stands by the light source and is asked, for example, "How strongly brighter is chair *B* than chair *C*?" He will then give one of the numbers from comparison described in the table and this judgment will be entered in the matrix on position (*B, C*). By convention, the comparison of strength is always of an activity appearing in the column on the left against an activity appearing in the row on the top. We then have the pairwise comparison matrix with four rows and four columns (a 4x4 matrix)

Brightness	*A*	*B*	*C*	*D*
A				
B				
C				
D				

The "agreed upon" numbers are the following. Given elements *A* and *B*; if

A and *B* are equally important, insert 1
A is weakly more important than *B*, insert 3
A is strongly more important than *B*, insert 5
A is demonstrably or very strongly more important than *B*, insert 7
A is absolutely more important than *B*, insert 9
in the position (*A, B*) where the row of *A* meets the column of *B*.

An element is equally important when compared with itself, so where the row of and column of *A* meet in position (*A, A*) insert 1. Thus the main diagonal of matrix must consist of 1's, Insert the appropriate reciprocal 1, 1/3, …., or 1/9 where the column *A* meets the row *B*, i.e., position (*B, A*) for the reverse comparison of *B* with *A*. The numbers 2, 4, 6, 8 and their reciprocals are used to facilitate compromise between slightly differing judgments. We also use rational numbers to form ratios from the above scale values when it is desired to force consistency on the entire matrix from a few judgments, i.e., a minimum of n-1.

In general, what we mean by being consistent is that when we have a basic amount of raw data, all other data can be logically deduced from it. In doing pairwise comparison to relate n activities so that each one is represented in the data at least once, we need n-1 pairwise comparison judgments. From them all other judgments can be deduced simply by using the following kind of relation: if activity A_1 is 3 times more dominant than activity A_2 and activity A_1 is 6 times more dominant than activity A_3 then $A_1 = 3A_2$ and $A_1 = 6A_3$. It should follow that $3A_2 = 6A_3$ or $A_2 = 2A_3$ and $A_3 = 1/2A_2$. If the numerical value of the judgment in the (2,3) position were different from 2 then the matrix would be inconsistent. This happens frequently and is not a disaster. Even if one has the whole real numbers to use for judgments, unless he occupies his attention methodologically to build up the judgments from n-1 basic ones, his numbers are not likely to be consistent. In addition, for most problems it is very difficult to identify n-1 judgments, which relate all activities, and of which one is absolutely certain.

In turns out that the consistency of a positive reciprocal matrix is equivalent to the requirement that its maximum eigenvalue λ_{max} should be equal to n. It is also possible to estimate the departure from consistency by the difference $\lambda_{max}-n$ divided by n-1. We note that $\lambda_{max} \geq n$ is always true. How bad our consistency may be in a given problem may be estimated by comparing our value of $(\lambda_{max}-n)/(n-1)$ with its value from randomly chosen judgments and corresponding reciprocals in the reverse positions in a matrix of the same size. We have a table for such entries on page 21 from which the figures may be taken. Consistency will be dealt with more precisely in later chapters.

Let us now return to our chair of brightness example. There are sixteen spaces in the matrix for our numbers. Of these, four are predetermined, namely, those in the diagonal, (*A, A*), (*B, B*), (*C, C*), (*D, D*), and have the value 1, since, for example, chair *A* has the same brightness as itself. Of the remaining twelve numbers, after the diagonal is filled in, we need to provide six, because the other six are reverse comparisons and must be reciprocals of the first six. Suppose the individual, using the recommended scale, enters the number 4 in the (*B, C*) position. He thinks chair *B* is between weakly and strongly brighter than chair *C*. Then the reciprocal value ¼ is automatically entered in the (*C, B*) position. It is not mandatory to enter a reciprocal, but it is generally rational to do so.

After the remaining five judgments have been provided and their reciprocals also entered, we obtain for the complete matrix

Brightness	*A*	*B*	*C*	*D*
A	1	5	6	7
B	1/5	1	4	6
C	1/6	1/4	1	4
D	1/7	1/6	1/4	1

The next step consists of the computation of a vector of priorities from the given matrix. In mathematical terms the principal eigenvector is computed, and when the normalized becomes the vector of priorities. We shall see in the next chapter that the relative brightness of the chairs expressed by this vector satisfies the inverse square law of optics. In the absence of a large scale of computer to solve the problem exactly, crude estimates of that vector can be obtained in the following four ways:

(1) *The crudest* Sum the elements in each row and normalize by dividing each sum by the total of all the sums, thus the results now add up to unity. The first entry of the resulting vector is the priority of the first activity; the second of the second activity and so on.
(2) *Better* Take the sum of the elements in each column and form the reciprocals of these sums. To normalize so that these numbers add to unity, divide each reciprocal by the sum of the reciprocals.
(3) *Good* Divide the elements of each column by the sum of that column (i.e., normalize the column) and then add the elements in each resulting row and divide this sum by the number of elements in the row. This is a process of averaging over the normalized columns.
(4) *Good* Multiply the *n* elements in each row and take the *n*th root. Normalize the resulting numbers.

A simple illustration which shows that methods (1), (2), and (3) produce the expected answer uses an urn with 3 white (*W*), 2 black (*B*), and red (*R*) balls. The probabilities of drawings a *W, B,* or *R* are, respectively, ½. 1/3. 1/6. It is easy to see that any of the first three methods gives these probabilities when applied to the following consistent pairwise comparison matrix. Method (4) also gives this result.

	W	*B*	*R*
W	1	3/2	3
B	2/3	1	2
R	1/3	1/2	1

It is important to note that these methods give different results for the general case where a matrix is not consistent.

Let us now apply the different methods of estimating the solution to the chair example.

Applying method (1), the sum of the rows of this matrix is a column vector which, to save space, we write as the row (19.00, 11.20, 5.42, 1.56). The total sum of the matrix is given by summing these vector components. Its value is 37.18. If we divide each components of the vector by this number we obtain the column vector of priorities, again written as row, (0.51, 0.30, 0.15, 0.04) for the relative brightness of chairs *A, B, C D*, respectively.

Applying method (2), the sum of the columns of this matrix is a row vector (1.51, 6.43, 11.25, 18.00). The reciprocals of these sums are (0.66, 0.16, 0.09, 0.06), which when normalized become (0.68, 0.16, 0.09, 0.06).

Applying method (3) we normalize each column (add its components and divide each component by this sum) obtaining the matrix

$$\begin{bmatrix} 0.66 & 0.78 & 0.53 & 0.39 \\ 0.13 & 0.6 & 0.36 & 0.33 \\ 0.11 & 0.04 & 0.09 & 0.22 \\ 0.09 & 0.03 & 0.02 & 0.66 \end{bmatrix}$$

The sum of the rows is the column vector (2.36, 0.98, 0.46, 0.20) which when averaged by the sample size of 4 columns yields the column vector of priorities (0.590, 0.254, 0.115, 0.050).

Method (4) gives (0.61, 0.24, 0.10, 0.04).

The exact solution to the problem, as will be described later in the book, is obtained by raising the matrix to arbitrarily large powers and dividing the sum of each row by the sum of the elements of the matrix. To two decimal places it is given by (0.61, 0.24, 0.10, 0.05).

By comparing these results we note that the accuracy is improved from (1) to (2) to (3), although they increase in complexity of computation. If the matrix were consistent all these four vectors would be the same. Method (4) only gives a very good approximation in the inconsistent case.

If we may assume that the reader knows how to multiply a matrix by vector, we can introduce a method for getting a crude estimate of consistency.

We multiply the matrix of comparisons on the right by the estimated solution vector obtaining a new vector. If we divide the first component of this vector by the first component of the estimated solution vector, the second component of the new vector by the second component of the estimated solution vector and so on, we obtain another vector. If we take the sum of the components of this vector and divide by the number of components we have an approximation to a number λ_{max} (called the maximum or principal eigenvalue) to use in estimating the consistency as reflected in the proportionality of preference. The closer λ_{max} is to n (the number of activities in the matrix) the more consistent is the result.

As will be clear from our theoretical discussion in a later chapter, deviation from consistency may be represented by $(\lambda_{max}-n)/(n-1)$ which we call the *consistency index* (C.I).

We shall call the consistency index of a randomly generated reciprocal matrix from the scale 1 to 9, with reciprocals forced, the *random index* (R.I). At Oak Ridge National Laboratory, colleagues (see Chap. 3) generated an average R.I for matrices of order 1-15 using a sample size of 100. One would expect the R.I. to increase as the order of the matrix increases. Since the sample size was only 100, there remained statistical fluctuations in the index from one order to another. Because of these, we repeated the calculations at the Wharton School for a sample size 500 up to 11 by 11 matrices and then used the Oak Ridge results for n = 12, 13, 14, 15. The following table gives the order of the matrix (first row) and the average R.I. (second row) determined as described above.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0.00	0.00	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.49	1.51	1.48	1.56	1.57	1.59

The ratio of C.I. to the average R.I. for the same matrix is called the *consistency ratio* (C.R.). A consistency ratio of 0.10 or less is considered acceptable.

To illustrate our approximate calculation of C.I. with an example we use the above matrix and the third column vector derived by method (3) to find λ_{max}. We had (0.59, 0.25, 0.11, 0.05) for the vector of priorities. If we multiply the matrix on the right by this vector we get the column vector (2.85, 1.11, 0.47, 0.20). If we divide corresponding components of the second vector by the first we get (4.83, 4.44, 4.28, 4.00). Summing over these components and taking the average gives 4.39.

This gives (4.39 – 4)/3 = 0.13 for the C.I. To determine how good this result is we divide it by the corresponding value R.I = 0.90. The consistency ratio (C.R.) is 0.13/0.90= 0.14 which is perhaps not as close as we would like to 0.10.

These comparisons and computations establish the priorities of the elements of one level of a hierarchy with respect to one element of the next level. If there are more than two levels, the various priority vectors can be combined into priority matrices, which yield one final priority vector for the bottom level.

1-5 INTUITIVE JUSTIFICATION OF THE METHOD

Assume that n activities are being considered by a group of interested people. We assume that the group's goals are:

(1) To provide judgments on the relative importance of these activities;
(2) To insure that the judgments are quantified to an extent which also permits a quantitative interpretation of the judgments among all activities.

Clearly, goal (2) will require appropriate technical assistance.

Our goal is to describe a method of deriving, from the group's quantified judgment (i.e., from the relative values associated with *pairs* of activities), a set of weights to be associated with *individual* activities; in a sense defined below, the weights should reflect the group's quantified judgments. What this approach achieves is to put the information resulting from (1) and (2) into usable form without deleting information residing in the qualitative judgments.

Let $C1$, $C2$, …., C_n be the set of activities. The quantified judgments on pairs of activities C_i, C_j are represented by an n-by-n matrix

$$A=(a_{ij}), \qquad (i, j = 1, 2, \ldots, n)$$

The entries a_{ij} are defined by the following entry rules.
Rule 1. If $a_{ij} = \alpha$, then $aij = 1/\alpha$, $\alpha \neq 0$.
Rule 2. If C_i is judged to be of equal relative importance as C_{ij}, then $a_{ij} = a_{ji} = 1$; in particular, $a_{ii} = 1$ for all i.

Thus the matrix A has the form

$$A = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 1/a_{12} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ 1/a_{1n} & 1/a_{2n} & \cdots & 1 \end{bmatrix}$$

Having recorded the quantified judgments on pairs (C_i, C_j) as numeric entries a_{ij} in the matrix A, the problem now is to assign to the n contingency C_1, C_2, …, C_n a set of numerical weights w_1, w_2, …., w_n that would "reflect the recorded judgments".

In order to do so, the vaguely formulated problem must first be transformed into a precise mathematical one. This essential, and apparently harmless, step is the most crucial one in any problem that requires the representation of a real-life situation in terms of an abstract mathematical structure. It is particularly crucial in the present problem where the representation involves a number of transitions that are not immediately discernible. It appears, therefore, desirable in the present problem to identify the major steps in the process of representation and to make each step as explicit as possible in order to enable the potential user to form his own judgment on the *meaning and value* of the method in relation to *his* problem and *his* goal.

The major question is the one concerned with the meaning of the vaguely formulated condition in the statement of our goal: "these weights should reflect the group's quantified judgments." This presents the need to describe in precise, arithmetic terms, how the weights, w_i should relate to the judgments a_{ij}: or, in other words, the problem of specifying the conditions we wish to impose on the weights we seek in relation to the judgments obtained. The desired description is developed in three steps, proceeding from the simplest special case to the general one.

Step 1 Assume first that the "judgments" are merely the result of precise physical measurements. Say the judges are given a set of pebbles, C_1, C_2, …, C_n and a precision scale. To compare C_1 with C_2, they put C_1 on a scale and read off its weight –say, w_1-305 grams. They weight C_2 and find $w_2 = 244$ grams. They divide w_1 by w_2, which is 1.25. They pronounce their judgment, "C_1 is 1.25 times as heavy as C_2 and record it as $a_{12} =$ 1.25.

Thus, in this ideal case of *exact measurement,* the relations between the weights w_1 and the judgments a_{ij} are simply given by

$$\frac{w_1}{w_j} = a_{ij} \qquad \text{(for } i, j = 1, 2, \ldots, n\text{)} \tag{1-1}$$

and

$$A = \begin{bmatrix} w_1/w_1 & w_1/w_1 & \cdots & w_1/w_n \\ w_1/w_1 & w_1/w_1 & \cdots & w_1/w_n \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ w_n/w_1 & w_n/w_2 & \cdots & w_n/w_n \end{bmatrix}$$

However, it would be unrealistic to require these relations to hold in the general case. Imposing these stringent relations would, in most practical cases, make the problem of finding the w_i (when a_{ij} are given) unsolvable. First, even physical measurements are never exact in a mathematical sense; and, hence, allowance *must* be made for deviations; and second, because in human judgments, these deviations are considerable larger.

Step 2 In order to see how to make allowance for deviations, consider the *i*th row in the matrix *A*. The entries in that row are

$$a_{i1}, a_{i2}, \ldots., a_{in}$$

In the ideal (exact) case these values are the same as the ratios

$$\frac{w_i}{w_1}, \frac{w_i}{w_2}, \cdots, \frac{w_i}{w_j}, \cdots, \frac{w_i}{w_n}$$

Hence, in the ideal case, if we multiply the first entry in that row by w_1, the second entry by w_2, and so on, we would obtain

$$\frac{w_i}{w_1} w_1 = w_i, \qquad \frac{w_i}{w_2} w_2 = w_i, \cdots, \frac{w_i}{w_j} w_j = w_i, \cdots, \frac{w_i}{w_n} w_n = w_i$$

The result is a row of identical entries

$$w_i, w_i, \ldots, w_i$$

whereas, in the general case, we would obtain a row of entries that represent a statistical scattering of values around w_i. It appears, therefore, reasonable to require that w_i should equal the average of these values. Consequently, instead of the ideal case relations (1-1)

$$w_i = a_{ij} w_j \qquad (i, j = 1, 1, \ldots, n)$$

the more realistic relations for the general case the form (for each fixed *i*)

$$w_i = \text{ the average of } (a_{i1} w_1 a_{i2}, \ldots, a_{in} w_n)$$

More explicitly we have

$$w_i = \frac{1}{n}\sum_{j=1}^{n} a_{ij} w_j \qquad (i = 1, 2, \ldots, n) \qquad (1\text{-}2)$$

While the relations in (1-2) represent a substantial relaxation of the more stringent relations (1-1), there still remains the question: is the relaxation sufficient to ensure the existence of solution; that is, to insure that the problem of finding a unique weights w_i when the a_{ij} are given is a solvable one?

Step 3 To seek the answer to the above essentially mathematical question, it is necessary to express the relations in (1-2) in still another, more familiar form. For this purpose we need to summarize the line of reasoning to this point. In seeking a set of conditions to describe how the weight vector *w* should relate to the quantified judgment, we first considered the ideal (exact) case in Step 1, which suggested the relations (1-1). Next, realizing that the real case will require allowances for deviations, we provided for such allowances in Step 2, leading to the formulation (1-2). Now, this is still not realistic enough; that is, that (1-2) which works for the ideal case is still too stringent to secure the existence of a weight vector *w* that should satisfy (1-2). We note that for good estimates a_{ij} tends to be close to w_i/w_j and hence is a small perturbation of this ratio. Now as a_{ij} changes it turns out that there would be corresponding solution of (1-2), (i.e., w_i and w_i can change to accommodate this change in a_{ij} from the ideal case), if *n* were also to change. We denote this value of *n* by $\lambda_{\max}$. Thus the problem

$$w_i = \frac{1}{\lambda_{\max}}\sum_{j=1}^{n} a_{ij} w_j \qquad i = 1, \ldots, n \qquad (1\text{-}3)$$

has a solution that also turns out to be unique. This is the well-known eigenvalue problem with which we will be dealing.

In general, deviations in the a_{ij} can lead to large deviations both in $\lambda_{\max}$ and $w_i, i = 1, \ldots, n$. However, this is not the case for a reciprocal matrix which satisfies rules 1 and 2. In this case we have a stable solution.

Recall that we have given an intuitive justification of our approach. There is an elegant way of framing this in mathematical notation. It is given in detail in later chapters. Briefly stated in matrix notation, we start with that we call the paradigm case $Aw = nw$, where *A* is a consistent matrix and consider a reciprocal matrix A^1 which is perturbation of *A*, elicited from pairwise comparison judgments, and solve the problem $A^1w^1 = \lambda_{max}w^1$ is the largest eigenvalue of A^1.

We have sometimes been interested in the opposite questions to dominance with respect to a given property. We have called it recessiveness of one activity when compared with another with respect to that property. In that case we solve for the left eigenvector *v* in $vA = \lambda_{\max}v$. Only when *A* is consistent are the elements of *v* and *w* reciprocals. Without consistency they are reciprocals for n=2 and n=3. In general one need not expect them to have a definite relationship. The two vectors correspond to the two sides of the Janus face of reality –the bright and the dark.

1-6 HIERARCHICAL COMPOSITION OF PRIORITIES BY EXAMPLE

School Selection Example

Three highshools, *A, B, C*, were analyzed from the standpoint of the author's son according to their desirability. Six independent characteristics were selected for the comparison –learning, friends, school life, vocational training, college preparation, and music classes (see Fig. 1-4). The pairwise judgment matrices were as shown in Table 1-2 and 1-3.

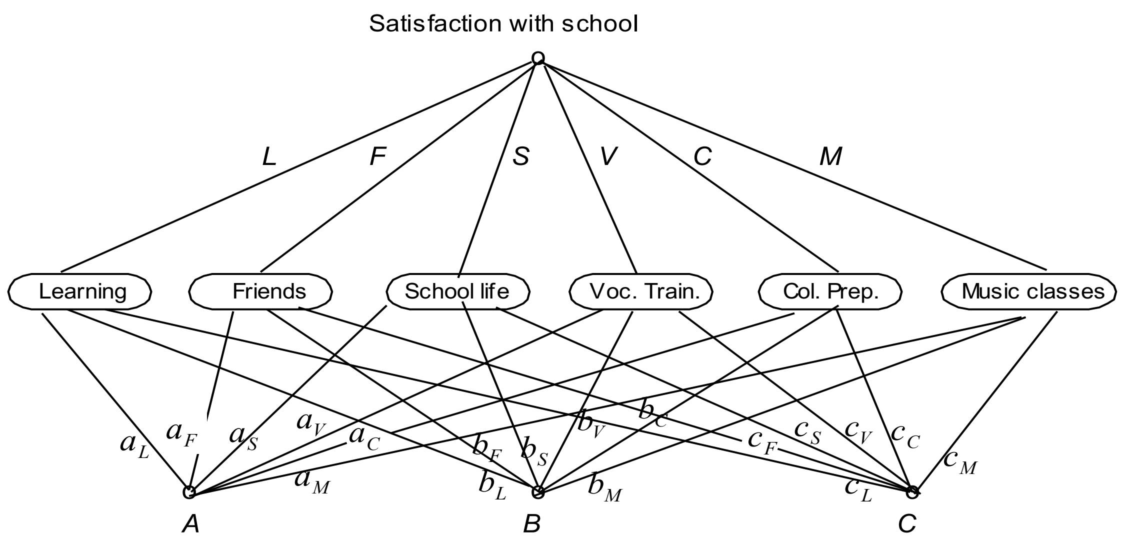

Figure 1-4 School satisfaction hierarchy

Table 1-2 Comparison of characteristics with respect to overall satisfaction with school

	Learning	Friends	School life	Vocational training	College preparation	Music classes
Learning	1	4	3	1	3	4
Friends	1/4	1	7	3	1/5	1
School life	1/3	1/7	1	1/5	1/5	1/6
Vocational training	1	1/3	5	1	1	1/3
College preparation	1/3	5	5	1	1	3
Music classes	1/4	1	6	3	1/3	1

$\lambda_{max} = 7.49,$ C.I. = 0.30, C.R. = 0.24

Table 1-3 Comparison of schools with respect to the six characteristics

Learning	*A*	*B*	*C*
A	1	1/3	1/2
B	3	1	3
C	2	1/3	1

λ_{max} = 3.05
C.I. = 0.025
C.R. = 0.04

Friends	*A*	*B*	*C*
A	1	1	1
B	1	1	1
C	1	1	1

λ_{max} = 3.00
C.I. = 0
C.R. = 0

School life	*A*	*B*	*C*
A	1	5	1
B	1/5	1	1/5
C	1	5	1

λ_{max} = 3.00
C.I. = 3.00
C.R. = 0

Vocational training	*A*	*B*	*C*
A	1	9	7
B	1/9	1	1/5
C	1/7	5	1

λ_{max} = 3.21
C.I. = 0.105
C.R. = 0.18

College preparation	*A*	*B*	*C*
A	1	1/2	1
B	2	1	2
C	1	1/2	1

λ_{max} = 3.00
C.I. = 0
C.R. = 0

Music classes	*A*	*B*	*C*
A	1	6	4
B	1/6	1	1/3
C	1/4	3	1

λ_{max} = 3.05
C.I. = 0.025
C.R. = 0.04

The priority vector of the first matrix is given by

(0.32, 0.14, 0.03. 0.13, 0.24,0.14)

and its corresponding eigenvalue is λ_{max}= 7.49 which is somewhat far from the consistent value 6. The C.I. is 0.30 and C.R. is 0.30/1.24 = 0.24, which is high.

To obtain the overall ranking of the schools, we multiply the last matrix on the right by the transpose (column version) of the row vector of weights of the characteristics. This is the same as weighting each of the above six eigenvectors by the priority of the corresponding characteristics and then adding (made possible by the independence of the characteristics (see below for further elaboration). This yields

$$A = 0.37$$
$$B = 0.38$$
$$C = 0.25$$

Table 1-4

Learning	Friends	School life	Vocational training	College preparation	Music classes
0.16	0.33	0.45	0.77	0.25	0.69
0.59	0.33	0.09	0.05	0.50	0.09
0.25	0.33	0.46	0.17	0.25	0.22

The son went to school *A* because it had the almost rank as school *B*, yet school *B* was a private school charging close to $1,600 a year and school *A* was free. This was a conflict problem between the author's son and wife; the first preferred school *A*, and the second school *B*, but neither took money into consideration as important. Although the C.R. for the second level was high they took the decision anyway despite protestations from author about high inconsistency.

Explanation using Fig. 1-4 If the weights of the criteria and the schools with respect to each criterion are as indicated along each line segment in the figure, then

Overall rank of school $A = a_L L + a_F F + a_s S + a_v V + a_c C + a_M M$
Overall rank of school $B = b_L L + b_F F + b_s S + b_v V + b_c C + b_M M$
Overall rank of school $C = c_L L + c_F F + c_s S + c_v V + c_c C + c_M M$

The previous calculations are the same as the following matrix multiplication

$$\begin{bmatrix} 0.16(a_L) & 0.33(a_F) & 0.45(a_S) & 0.77(a_V) & 0.25(a_C) & 0.69(a_M) \\ 0.59(b_L) & 0.33(b_F) & 0.09(b_S) & 0.05(b_V) & 0.50(b_C) & 0.09(b_M) \\ 0.25(c_L) & 0.33(c_F) & 0.46(c_S) & 0.17(c_V) & 0.25(c_C) & 0.22(c_M) \end{bmatrix} \begin{bmatrix} 0.32(L) \\ 0.14(F) \\ 0.03(S) \\ 0.13(V) \\ 0.24(C) \\ 0.14(M) \end{bmatrix}$$

To find out the measure of satisfaction of a candidate with a school, first we need to list the important criteria which characterize schools and compute the relative desirabilities of these criteria to the candidate. Desirability would vary from one candidate to another. For example, one student may find friends more attractive than college preparation while another may feel the opposite way. The criteria are denoted by *L, F, S, V, C,* and *M* in the figure.

The second step is to compute the relative standing of each school with respect to each criterion. For example, one school may have better music classes while another is well known for its vocational training.

To get the overall ranking of each school, first we need to multiply the weight indicating the qualification of that school with respect to the criterion by the weight of that criterion. We then add these values for each school with respect to all the criteria. Since the relative weight of learning is L, a_L is the overall weight of learning for school *A*. By the same argument we calculate $a_F F$, $a_s S$, $a_V V$, $a_C C$, $a_M M$. Therefore, the overall rank of school *A* is the sum of the overall weights of the activities mentioned previously, i.e., overall rank of school $A = a_L L + a_F F + a_s S + a_V V + a_C C + a_M M$.

The reader who is interested in the perversity of youthful judgment may wish to see what the priorities look like three years later (Table 1-5). The young man (now aged 18) no longer considers friends or vocational training as important. His interest in college and music seem to dominate. They have become urgent need rather than long range aspirations. Consistency has also improved tremendously.

The priorities of the schools with respect to the characteristics are the same as before and it is now much clearer than the right choice was made then. The priorities of the schools are *A*=0.40, *B*=0.36, *C*=0.25.

1-7 PROTOCOL OF A PRIORITIZION SESSION

The first requirement in the analysis of the functions of a system is to construct the hierarchy representing these functional relations. So far it has been found that for most simple systems, the hierarchy suggests itself in a natural correspondence with the functions of the system. However, the system may have a high degree of complexity and it may not be easy to find the hierarchical structure which corresponds to this system. In a more direct approach we have often resorted to the process of brainstorming, by putting down all elements relevant to that hierarchy. We then arranged these in groups according to dominance among the groups. These groupings served as the hierarchy levels. This process of groupings can be better accomplished by a technical procedure described later. It may be useful to mention that two properties of a hierarchy level which have strong overlap should be grouped together as a single more general property for the comparison. For example, quality and size often go together and may be grouped together as suitability.

Table 1-5 Overall satisfaction with school

	Learning	Friends	School life	Vocational training	College preparation	Music classes
Learning	1	5	7	5	3	1
Friends	1/5	1	3	1/5	1/6	1/6
School life	1/7	1/3	1	1/4	1/5	1/5
Vocational training	1/5	5	4	1	1/5	1/6
College preparation	1/3	6	5	5	1	1
Music classes	1	6	5	6	1	1

The eigenvalue of this matrix is λ_{max}= 6.68, C.I. = 0.14, C.R. = 0.14
The corresponding eigenvector is (0.33, 0.05, 0.03, 0.09, 0.23, 0.27)

To assist with the quality of informed judgment inputs, it is essential that the hierarchy of activities, objectives, and still higher objectives be set up with care. A study may be required to identify and characterize those properties in the levels of the hierarchy which affect the performance of the higher-level properties or the fulfillment of higher-level objectives.

After dividing the ideas into categories, the process of defining the purposes for which the problem is being studied and structuring the hierarchy is carefully and methodically carried out. Tentativeness in the structuring process is essential. What is most important is that an individual's knowledge and judgments or those of a group have a fair chance of being adequately and correctly expressed. This is not a task for a weary and short-tempered director. Diplomacy and concern for the feelings of the people involved are paramount. Yet the leader himself must make sure that differences do not cause the process of deteriorate. Occasionally it helps to remind the participants that someone has to do something about the problem, and that if they are not able to crystallize their ideas, the result may come out contrary to what they might desire to see happen in a fair process.

Before proceeding with prioritization, we urge that an attempt be made to write down a definition of the elements introduced to avoid controversial arguments later on.

The same approach may be used to assist a single decision maker in organizing the complexity he faces and derive priorities which reflect his belief and attitudes. In a complex situation there is little hope that problems can be resolved by an internalized mystical, but not articulated, understanding of the important factors. It may be counterproductive to be perpetually concerned that the process may leave out some important factors. If an individual really understands the process he would have to be aware of the important factors and keep examining his feelings for residual factors that are important but not yet included. This is one reason why one should take time to study a problem and not rush through it.

The quality of the output may be evaluated by how logically satisfactory the answers are. They must, in some sense, conform with the original input. For example, a member of a level that is favored over the other members through the original pairwise judgments should come out with the highest ranking and so on down the line. Or course, it is the very purpose of the model to develop a consistent order. Note that the total ordering is *not* known at the beginning, but only pairwise comparisons which may in fact be inconsistent. The results must conform with what one intuitively expects as a reasonable outcome. Otherwise, there would be discrepancy between the judgments provided and the operations of the theory.

It is important to remark that the numbers used in the scale are absolute magnitudes and not simple ordinal numbers. This says that our scale does not allow comparisons whose intensity exceeds 9. As we have indicated, elements must be put into clusters within each of which the elements are comparable with this scale, and then the clusters in turn must also be compared with this scale. Note that it may be necessary to invent or introduce intermediate clusters in order to be able to make relative comparisons which lead from the cluster with the smallest (or weakest) elements to be cluster with the largest (or strongest) elements. This is the natural way we do things and not an artifice adopted for the theory. We cannot directly compare the weight of a grain of sand with that of the sun. We need a gradual transition between them.

One must prioritize very carefully the highest levels of the hierarchy because it is there that consensus is most needed since these priorities drive the rest of the hierarchy. In each level one must ensure that the criteria represented are independent or at least are sufficiently different, and that these differences can be captured as independent properties in the level. Revision of the elements may be necessary to capture independence successfully. Since there are times when dependence is holistic and cannot be removed, our approach can also be adapted to handle interdependence, as we shall see in Chap. 6. As one goes farther down the hierarchy one expects greater variability of opinion among compatible people as we reach the operating level. In that case, each person wants a piece of the action. To the extent that people agree about the meaning as well as the importance of the elements more resources should be allocated to that area; to the extent that people disagree about the either meaning or importance, their judgments tend to nullify each other and the area tends to get smaller share of the action until greater support for it is obtained. If an area important to our needs, but there is disagreement on implementation, we would have to withhold action until people develop a better appreciation for the need and can induce more cohesive action. This is a logic outcome of the hierarchical approach. Where there is disagreement, people will tend to be dissatisfied because they don't see realization of their judgments. Otherwise, with agreement there is greater satisfaction.

A large audience of diverse backgrounds would require a great deal of time to structure a hierarchy and provide judgments. Wear and tear may set in early and the meeting may not lead to fruitful results in the allotted time. The best way to engage a large group is either to choose a narrow focus for the discussion or better, to have them generate the hierarchy (or provide them with one for debate) and then divide them into homogenous groups and let each group provide the judgments in those parts of the hierarchy which relate to their special interest. People should be told that some might feel frustrated during the process; they could go out for a walk or participate in a discussion in a separate room while the others carry on, and then return when they feel reinvigorated. This avoids deterioration of the process.

Of course there are times when political favors, hidden agenda, disruption, and other political processes may be in operation and group interaction and cooperation would be difficult. We have encountered such problems in our experience using the Analytic Hierarchy Process (AHP). Our conclusion is that the AHP is a powerful tool for those who want to assess their own and their opponent's strategies. Those who do not wish to participate cannot be forced to, but they can sometimes coaxed to do so.

In cooperative undertakings, the process moves faster when the participants have in common: (1) shared goals; (2) intimate long term contact; (3) work in a climate of social acceptance; and (4) have equal status when participating.

A final observation is that group interaction is not unlike a marriage, about which people tend to have romantic feelings at the start, but as they get into it they find that there is a good deal of friction, feuding, and dissent. However, overall, life moves on and there are fundamental points of agreement and mutual needs which keep people together. Thus one must not enter any group interaction process with too much idealism and a strong predisposition for propriety and order.

We now turn our attention to the next step in the process, which is to solicit informed judgments from people.

We are given the elements of a hierarchy level and wish to construct the matrix of pairwise comparison among these elements in relation to each element of the next higher level which serves as a criterion or property with respect to which they are compared. The individuals, who give the judgments, are asked the following type of question: Given a pair of elements of the matrix, which one do you believe in more dominant in possessing or contributing to the property in question? How strong is this dominance: equal, weak, strong, demonstrated, or absolute, or is it a compromise between adjacent values in this strength comparison?

The question must be carefully phrased to evoke the judgment or feeling of the individuals involved. Uniformity should be maintained in the questions asked. It is essential to focus on the property involved as people's minds may wander fuzzily to more general properties.

Remark In order to obtain a set of priorities that reflect the merit or positive impact of the activities, the set of properties with respect to which they are compared must be formulated in such a way that the desirable attributes of the activities are brought out. For example, the cost of going on vacation would yield a high priority number for the more expensive vacation spot, but in fact, this priority should be low. In that case, the question to ask is: Which vacation place *saves* more on cost rather than which one costs more?

If the individual differ in their judgment, they are allowed to make a case for themselves by either reaching consensus (which sometimes happens even after a heated debate) or by following whether ground rules there are for reaching a single judgment, such as majority vote. Individuals have been known to change their position. In some cases a whole group changed their position after listening to reasons given by one member. Bargaining is possible whereby people accept the judgments given by others in return for using their judgment in another area more important to them.

When people are reluctant to volunteer their judgment, an auction-type procedure may be followed by proposing a judgment value and asking people how they feel about it. Lack of inclination to discriminate between two elements often means that they share the property equally among them. When there is no agreement, each individual records his judgments and the solutions are examined for a clearer understanding of what (if anything) can be done. There are times when differences in the world of people cripple action.

When the entire set of judgments has been obtained, people are asked about how faithfully they feel their understanding and judgments have been represented. This avoids hard feelings arising out of being ignored. Debates might be shortened if time is limited, but people should be reminded that it is their problem and requires sufficient time to get good results. The participants should always be consulted about the adequacy of the hierarchical structuring of their problem and the representation of their judgments. If there are objections, they should be carefully and patiently considered. If revisions are desired, they may be assigned as subtasks to be performed soon and reported on to the group.

Frequently, one can note areas of greatest difference in judgment and bring them up again later in the session for review.

The procedure may begin by focusing on the rows of the matrix in the order of believed dominance of their corresponding elements essentially implying that people can probably tell the ordinal dominance of the elements in advance. The strongest and weakest elements are compared first to provide a guidepost for the other values. Of course, this may no be always possible. Another way is to try and find out those comparisons which people are sure of.

The numerical values and their reciprocals are entered in the matrix each time a judgment is obtained and soon people learn to give the numbers directly. The geometric means of the judgments may be used when people don't want to enter into debate. This is probably a less desirable alternative. Sometimes one can obtain the individual priority vectors and take their geometric mean for an answer.

It is worth noting that at times lower priority criteria finally determine the choice of alternatives. Consider an average family of four buying a car. The more important criterion is the budget (priority 0.52) they have available. Next is the price of the car (priority 0.23). A relatively low priority is the style and size (priority 0.16) and, finally, economy of operation (priority 0.09). Once they have selected several cars of the same price range allowed by their budget, the final selection one of this group is dictated by the style and economy. The higher priority criteria help in choosing the suitable and affordable class of car; the lower priority criteria help in choosing the individual car from among the brands.

Four types of questions are sometimes raised with regard to the judgments process: (1) the primary effect, or whether providing judgments may not bias the outcome toward what is examined first; (2) the recency effect, or the influence of the latest information over what went before, (3) the out-of-role- behavior where people assume the role of others and provide judgments for them without full appreciation of the people they represent; and (4) personal bias while participating in group decision making. Most of these phenomena can occur in an ordinary group session. Their influence is diminished if more time is taken with repeated interaction and people are cautioned about personal bias. In other words to correct problems of handing information different repetitions of the problem should highlight these difficulties leading to a final exercise considered by the group to be representative of the problem.

For the Decision-maker

If you are faced with a number of options to choose from and you have a maze of criteria to judge with, do the pairwise comparison of the criteria with respect to *short* and *long*-range efforts, *risks* and *benefits*, and also make a pairwise comparison matrix with respect to effectiveness and success. Finally, on the lowest level, compare the options with respect to each criterion, compose the weights hierarchically, and select the highest priority. If you have canvassed enough judgments so that you are sure you have considered all the relevant factors and good judgments, stop agonizing over your choice. You have done your human best to make the right choice. For quick decisions in day-to-day operations maintain a file of your working hierarchies, their judgments and resulting priorities. Change the necessary judgments for that decision to obtain the result or note which judgments have to be changed to obtain a desired result. Finally, add elements with their relevant judgments if necessary to obtain new priorities. This can also be done by interacting with a computer which has the information stored. For portfolio selection, a benefits hierarchy and a costs hierarchy are needed. The ratios of benefits to costs are then used for decision purposes.

1-8 SUMMARY

The eigenvalue approach to pairwise comparisons provides a way for calibrating a numerical scale, particularly in new areas where measurements and quantitative comparisons do not exist. The measure of consistency enables one to return to the judgments modifying them here and there to improve the overall consistency. The participation of several people makes it possible to make tradeoffs between different entries. It can also create a dialogue for what the real relation should be: a compromise among the various judgments representing diverse experience.

The steps of the process proceed as follows.

(1) State the problem.
(2) Put the problem in broad context –embed it if necessary in a larger system including other actors, their objectives, and outcomes.
(3) Identify the criteria that influence the behavior of the problem
(4) Structure a hierarchy of the criteria, sub-criteria, properties of alternative, and the alternative themselves.
(5) In a many party problem the levels may relate to the environment, actors, actor objectives, actor policies, and outcomes, from which one derives the composite outcome (state of the world).
(6) To remove ambiguity carefully defines every element in the hierarchy.
(7) Prioritize the primary criteria with respect to their impact on the overall objective called the focus.
(8) State the question for pairwise comparisons clearly above each matrix. Pay attention to the orientation of each question, e.g., costs go down, benefits go up.
(9) Prioritize the sub criteria with respect to their criteria.
(10) Enter pairwise comparison judgments and force their reciprocals.
(11) Calculate priorities by adding the elements of each column and dividing each entry by the total of the column. Average over the rows of the resulting matrix and you have the priority vector.

For (12)-(15) see later chapters

(12) In the case of scenarios calibrate their state variables on a scale of –8 to 8- as to how they differ from the present zero.
(13) Compose the weights in the hierarchy to obtain composite priorities and also the composite values of the state variables which collectively define the composite outcome.

(14) In case of choosing among alternatives select the highest priority alternative.
(15) In the case of resource allocation, cost out alternatives, compute benefit to cost ratio and allocate accordingly, either fully or proportionately. In a cost prioritization problem allocate resources proportionately to the priorities.

1-9 HIERARCHIES AND JUDGMENTS BY QUESTIONNAIRE

It is possible to elicit the hierarchy concerning an issue by questionnaire, synthesize the result, and follow up by another questionnaire to elicit judgments.

We give a simple illustration of how judgments may be obtained for a single matrix by using a questionnaire. The same method can be applied to a hierarchy. Let us consider the optics example to obtain judgments on the relative brightness of chairs. We indicate scale values ranging from one extreme down towards equality and then again raising to the extreme. In a left column we list all the alternatives to be compared for dominance with other alternatives in the right column. In all, each column contains $[n(n-1)]/2$ alternatives. We then ask people to check the judgment, which indicates the dominance of the element in the left column over the corresponding one in its row in the right column. If in fact there is such dominance some position in the set of values to the left of equality is checked. Otherwise equality or a position in the right set of values is checked. The same is done for all alternatives.

Relative brightness

Column I	Absolute	Very strong	Strong	Weak	Equal	Weak	Strong	Very strong	Absolute	Column II
C_1	____	____	____	____	____	____	____	____	____	C_2
C_1	____	____	____	____	____	____	____	____	____	C_3
C_1	____	____	____	____	____	____	____	____	____	C_4
C_2	____	____	____	____	____	____	____	____	____	C_3
C_2	____	____	____	____	____	____	____	____	____	C_4
C_3	____	____	____	____	____	____	____	____	____	C_4