CHAPTER TWO

INSTRUCTIVE EXAMPLES

2-1 INTRODUCTION

In this chapter, we shall develop our method further primarily with the help of examples. First, we shall relate the "illuminated chairs" experiment and show that the relative brightness of the chairs, as determined by the subjective pairwise comparisons, are very close to those predicted by the inverse-square law of optics. As a further indication that our method produces, in cases where the actual figures are known, a close approximation to these values, we shall reproduce the results of the elementary study of the influence of nations through their wealth. Following that is an example estimating the relative distance of six cities from Philadelphia. We then distinguish between complete and incomplete hierarchies.

We close the chapter with two further examples. They were chosen in order to demonstrate how one determines an overall priority of the bottom level elements in a hierarchy with more than two levels. The first one gives us the opportunity to make some observations of more general interest.

2-2 TEST FOR ACCURACY, RMS AND MAD

Of considerable interest to us must be the issue of how closely the priority vector developed by our method matches the "real" priority sector. One way to ascertain this is to apply the method to situations which allow determination of the actual numbers. In such cases, we wish to check how accurate the priority vector is.

To test for accuracy we must compare estimates in experiments with real answers that are known. Comparison of numbers involves the use of statistical measures. There are not many measures for validating theoretical results against reality. Two are the root mean square deviation and the median absolute deviation about the median. They are usually used for comparison purposes among several sample estimates to choose the one closest to reality and not as absolute measures. Both are a means of measuring the spread of a set of measurements from a known set of underlying values.

Deviations between small numbers are apt to be small. To see how significantly small they are in absolute terms they must be divided by the average size number they are taken from. In our case it would be $1/n$ where n is the number of items being compared. Incidentally, one measure of error might be to take the differences (or absolute differences), weight them by the priorities, take their average, then divide by $1/n$, i.e., use $\sum_{i=1}^{n} w_i |w_i - x_i|$ where w_i are the priorities and x_i are their estimates.

The root mean square deviation (RMS) of two sets of numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ is: $\sqrt{\frac{1}{n}\sum_{i=1}^{n}(a_i - b_i)^2}$

The median of a set of n numbers is obtained by arranging the numbers in increasing order and taking the middle term if n is odd and the average of the two middle terms if n is even. The median absolute deviation about the median (MAD) of a set of numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ is given by median $\{|(a_i - b_i) - \text{median}(a_i - b_i)|\}$. As an illustration, see the illumination intensity example in the next section.

2-3 ILLUMINATION INTENSITY AND THE INVERSE SQUARE LAW

In Chap. 1 we presented the chair brightness example and proceeded as far as filling in the judgments and solving for the relative brightness. Four identical chairs were placed on a line from a light source at the distances of 9, 15, 21, and 28 yards. The purpose was to see if one could stand by the light and look at the chair and compare their relative brightness in pairs, fill in the judgment matrix and obtain a relationship between the chairs and their distance from the light source. This experiment was repeated twice with different judges whose judgment matrices we now give. The first of these was given in Chap. 1.

Relative visual brightness
(1st Trial)

	C_1	C_2	C_3	C_4
C_1	1	5	6	7
C_2	1/5	1	4	6
C_3	1/6	1/4	1	4
C_4	1/7	1/6	1/4	1

Relative visual brightness
(2nd Trial)

	C_1	C_2	C_3	C_4
C_1	1	4	6	7
C_2	1/4	1	3	4
C_3	1/6	1/3	1	2
C_4	1/7	1/4	1/2	1

The judges of the first matrix were the author's young children, ages 5 and 7 at that time, who gave their judgments qualitatively. The judge of the second matrix was the author's wife, who was not present during the children's judgment process.

Relative brightness eigenvector (1st Trial)	Relative brightness eigenvector (2nd Trial)
0.61	0.62
0.24	0.22
0.10	0.10
0.05	0.06

$\lambda_{max} = 4.39$
C.I. = 0.13
C.R.= 0.14

$\lambda_{max} = 4.1$
C.I. = 0.03
C.R.= 0.03

Table 2-1 Inverse square law of optics

Distance	Normalized distance	Square of normalized distance	Reciprocal of previous column	Normalized reciprocal	Rounding off
9	0.123	0.015 129	66.098	0.607 9	0.61
15	0.205	0.042 025	23.79	0.218 8	0.22
21	0.288	0.082 944	12.05	0.110 8	0.11
28	0.384	0.147 456	6.78	0.062 3	0.06

First and second trial eigenvectors should be compared with the last column of the Inverse Square Law Table (2-1) calculated from the inverse square law in optics. It is interesting and important to observe that the judgments have captured a natural law here. It would seem that they could do the same in other areas of perception or thought, as we shall see later.

Note that sensitivity of the results as the object is very close to the source, for then it absorbs most of the value of the relative index and a small error in its distance from the source yields great error in the values. What is noteworthy from this sensory experiment is the observation or hypothesis that the observed intensity of illumination varies (approximately) inversely with the square of the distance. The more carefully designed the experiment, the better the results obtained from the visual observations.

The RMS of (0.62, 0.22, 0.10, 0.06) and (0.61, 0.22, 0.11, 0.06) is $\{1/4[(0.01)^2 + 0 + (0.01)^2+0]\}^{1/2} = 2.23\text{x}10^{-3}$. The MAD is as follows. The differences between the two vectors are given by (0.01, 0, –0.01, 0). The median of these numbers is 0+0/2 = 0. The deviations about this median are (0.01, 0, –0.01, 0). Their absolute value is taken and the median of the result is (0.01+0)/2 = 0.005 = $5\text{x}10^{-3}$. The significance of both RMS and MAD may be determined by dividing their values by the average value of the vector components which is simply $1/n$, where n is the number of components. Two vectors are nearly the same if either or both ratios are, for example, less than 0.1.

2-4 WEALTH OF NATIONS THROUGH THEIR WORLD INFLUENCE (Saaty and Khouja, 1976)

A number of people have studied the problem of measuring world influence of nations. We have briefly examined this concept within the framework of our model. We assumed that influence is a function of several factors. We considered five such factors: (1) human resources; (2) wealth; (3) trade; (4) technology; and (5) military power. Culture and ideology, and potential natural resources (such as oil) were not included.

Seven countries were selected for this analysis. They are the U.S., U.S.S.R., China, France, U.K., Japan, and West Germany. It was felt that these nations as a group comprised of a dominant class of influential nations. It was desired to compare them among themselves as to their overall influence in international relations. We realize that what we have is a very rough estimate, mainly intended to serve as an interesting example of an application of our approach to priorities. We will only illustrate the method with respect to the single factor of wealth. The more general problem is studied in the paper referenced above.

In Table 2-2, we give a matrix indicating the pairwise comparisons of the seven countries with respect to wealth. For example, the value 4 in the first row indicates the wealth is between weak and strong importance in favor of the U.S. over the U.S.S.R. The reciprocal of 4 appears in the symmetric position, indicating the inverse relation of relative strength of the wealth of the U.S.S.R. compared to the U.S.

Table 2-2 Wealth

	U.S.	U.S.S.R.	China	France	U.K.	Japan	W.Germany
U.S.	1	4	9	6	6	5	5
U.S.S.R.	0.25	1	7	5	5	3	4
China	0.11	0.14	1	0.2	0.2	0.14	0.2
France	0.17	0.2	5	1	1	0.33	0.33
U.K.	0.17	0.2	5	1	1	0.33	0.33
Japan	0.2	0.33	7	3	3	1	2
W.Germany	0.2	0.25	5	3	3	0.5	1

$\lambda_{max} = 7.068$, C.I. = 0.10, C.R. = 0.08

Explanation of Table

The first row compares the wealth influence (e.g., the Marshall Plan, A.I.D., etc.) of the U.S. with the other nations. For example, it is of equal importance to the U.S. (hence, the unit entry in the first position), between weak and strong importance when compared with the U.S.S.R. (hence, the value 4 in the second position), of absolute importance when compared with China (hence, the value 9 in the third position). We have values between strong and demonstrated importance when compared with France and U.K. (hence a 6 in the next two positions), strong importance when compared with Japan and Germany (hence, a 5 in the following two positions). For the entries in the first column we have the reciprocals of the numbers in the first row indicating the inverse relation of relative strength of the wealth of the other countries when compared with the U.S. and so on for the remaining values in the second row and second column, etc.

Table 2-3 Normalized wealth eigenvector

	Normalized eigenvector	Actual GNP* (1972)	Fraction of GNP Total
U.S.	0.427	1,167	0.413
U.S.S.R.	0.230	635	0.225
China	0.021	120	0.043
France	0.052	196	0.069
U.K.	0.052	154	0.055
Japan	0.123	294	0.104
W.Germany	0.094	257	0.091

Note : Root Mean Square Deviation = 0.024
Estimates of the GNP of China range from 74 billion to 128 billion. Those of Russia are also uncertain.
* Billions of dollars

Note that the comparisons are not consistent. For example, U.S.: U.S.S.R. = 4, U.S.S.R.: China = 7 but U.S.: China = 9 (not 28). Nevertheless, when the requisite computations are performed, we obtain relative weights of 0.427 and 0.230 for the U.S. and Russia, and these weights are in striking agreement with the corresponding Gross National Products (GNP) as percentages of the total GNP (see Table 2-3).

Thus, despite the apparent arbitrariness of the scale, the irregularities disappear and the numbers occur in good accord with the observed data. Thus wealth influence is proportional to actual wealth.

Compare the normalized eigenvector column derived by using the matrix of judgments in Table 2-1 with the actual GNP fraction given in the last column. The two are very close in their values. Estimates of the actual GNP of China range from 74 billion to 128 billion.

The value for China is more than it is for Japan in that our estimate is half the (admitted uncertain) GNP value. Japan's value is a third over the true value. China probably does not belong in this group of nations.

2-5 ESTIMATING DISTANCES

Six cities were chosen: Montreal, Chicago, San Francisco, London, Cairo, and Tokyo. Their distances from Philadelphia were compared, pairwise by an experienced air traveler, who thought only of the airplane boredom and did not think of actual times or distances. The distance comparison matrix shown gives the judgments. The other matrix gives the actual distances, their normalized values, and the eigenvector derived from the judgment matrix.

Comparison of distances of cities from Philadelphia	Cairo	Tokyo	Chicago	San Francisco	London	Montreal
Cairo	1	1/3	8	3	3	7
Tokyo	3	1	9	3	3	9
Chicago	1/8	1/9	1	1/6	1/5	2
San Francisco	1/3	1/3	6	1	1/3	6
London	1/3	1/3	5	3	1	6
Montreal	1/7	1/9	1/2	1/6	1/6	1

λ_{max} = 6.45, C.I. = 0.09, C.R. = 0.07

City	Distance to Philadelphia (miles)	Normalized distance	Eigenvector
Cairo	5 729	0.278	0.263
Tokyo	7 449	0.361	0.397
Chicago	660	0.032	0.033
San Francisco	2 732	0.132	0.116
London	3 658	0.177	0.164
Montreal	400	0.019	0.027

2-6 TYPICAL HIERARCHIES

Figures 2-1 and 2-2 are illustrations of two different hierarchies.

In Fig. 2-1 the first hierarchy level has a single objective; the overall welfare of a nation. Its priority value is assumed to be equal to unity. The second hierarchy level has three objectives: strong economy, health, and national defense.

Their priorities are derived from a matrix of pairwise comparisons with respect to their objective of the first level. The third hierarchy level objectives are industries. The object is to determine the impact of the industries on the overall welfare of a nation through the intermediate second level. Thus their priorities with respect to each objective in the second level are obtained from a pairwise comparison matrix with respect to that objective, and the resulting three priority vectors are then weighted by the priority vector of the second level to obtain the desired composite vector of priorities of the industries.

In Fig. 2-2, the hierarchy consists of four levels, the first being the overall welfare of a nation, the second a set of possible future scenarios of that nation, the third level the provinces of that nation, and the fourth are transport projects which are to be implemented in the provinces. Note that not every province affects each scenario nor does each project affect every province. The hierarchy in Fig. 2-2 is not a complete one. The object is to determine the priorities of the projects as the impact on the overall objective. Here one must weigh the priorities of each comparison set by the ratio of the number of elements in that set to the total number of elements in the fourth level. This is done occasionally when the hierarchy is not complete. Sometimes an incomplete hierarchy may be studied as a complete hierarchy but using zeros for the judgments and their reciprocals in the appropriate place.

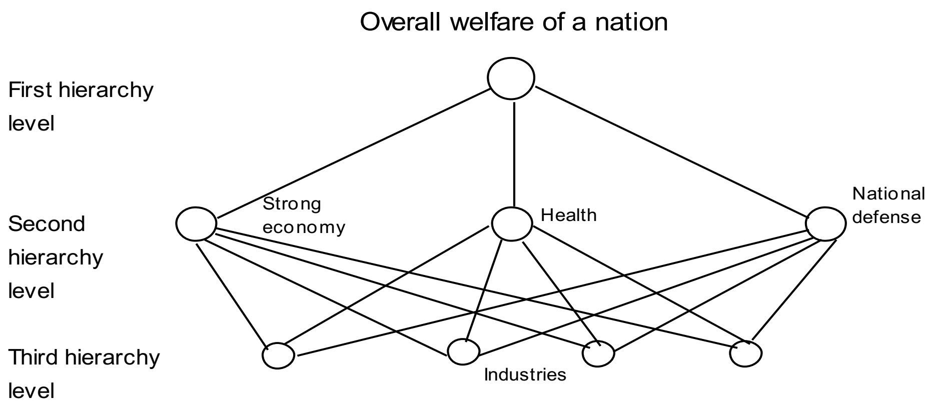

Figure 2-1 A complete hierarchy for priorities of industries

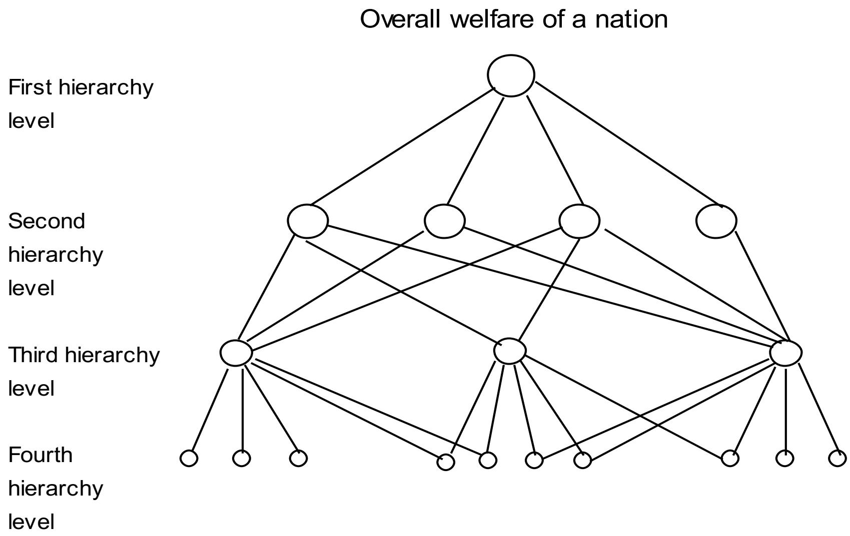

Figure 2-2 A hierarchy for priorities of transport projects in national planning

2-7 PSYCHOTERAPY

The Analytic Hierarchy Process may be used to provide insight into psychological problem areas in the following manner. Consider an individual's overall well being as the single top-level entry in a hierarchy. Conceivably this level is primarily affected by childhood, adolescent, and adult experiences. Factors in growth and maturity which impinge upon well-being may be the influences of the father and the mother separately as well as their influences together as parents; the socio-economic background; sibling relationships, one's peer group, schooling, religion status, and so on.

The above factors which comprise the second level in our hierarchy are further affected by criteria pertinent to each. For example, the influence of the father may be broken down to include his temperament, strictness, care, and affection. Sibling relationships can be further characterized by the number, age differential, and sexes of siblings; peer pressure and role modeling provide a still clearer picture of the effects of friends, schooling, and teachers.

As an alternative framework of description for the second level, we might include self-respect, security, adaptability to new people and new circumstances, and so on, influencing or as influenced by the elements above.

A more complete setting for a psychological history might include several hundreds of elements at each level, chosen by trained individuals and placed in such a way as to derive the maximum understanding of the subject in question.

Here we will consider a highly restricted form of the above, where the individual in question feels his self-confidence has been severely undermined and his social adjustments impaired by a restrictive situation during childhood. He is questioned about his childhood experiences only and asked to relate the following elements pairwise on each level.

Level 1. Overall well being (*OW*)
Level 2. Self-respect, sense of security, ability to adapt to others (*R, S, A*)
Level 3. Visible affection shown for subject (*V*)
Ideas of strictness, ethics (*E*)
Actual disciplining of child (*D*)
Emphasis on personal adjustments with others (*O*)
Level 4. Influence of mother, father, both (*M, F, B*)

The replies in the matrix form were as follows.

OW

	R	*S*	*A*
R	1	6	4
S	1/6	1	3
A	1/4	1/3	1

λ_{max} = 3.26
C.I. = 0.07
C.R. = 0.12

R

	V	*E*	*D*	*O*
V	1	6	6	3
E	1/6	1	4	3
D	1/6	1/4	1	1/2
O	1/3	1/3	2	1

λ_{max} = 4.35
C.I. = 0.12
C.R. = 0.13

S

	V	*E*	*D*	*O*
V	1	6	6	3
E	1/6	1	4	3
D	1/6	1/4	1	1/2
O	1/3	1/3	2	1

λ_{max} = 4.35
C.I. = 0.12
C.R. = 0.13

A

	V	*E*	*D*	*O*
V	1	1/5	1/3	1
E	5	1	4	1/5
D	3	1/4	1	1/4
O	1	5	4	1

λ_{max} = 5.42
C.I. = 0.47
C.R. = 0.52

V

	M	*F*	*B*
M	1	9	4
F	1/9	1	8
B	1/4	1/8	1

λ_{max} = 4.00
C.I. = 0.33
C.R. = 0.57

E

	M	*F*	*B*
M	1	1	1
F	1	1	1
B	1	1	1

λ_{max} = 3.00
C.I. = 0.00
C.R. = 0.00

D

	M	*F*	*B*
M	1	9	6
F	1/9	1	1/4
B	1/6	4	1

λ_{max} = 3.11
C.I. = 0.06
C.R. = 0.10

O

	M	*F*	*B*
M	1	5	5
F	1/5	1	1/3
B	1/5	3	1

λ_{max} = 3.14
C.I. = 0.07
C.R. = 0.12

The eigenvector of the first matrix, *a*, is given by

$$\begin{array}{c} \\ R \\ S \\ A \end{array} \begin{bmatrix} OW \\ 0.701 \\ 0.193 \\ 0.106 \end{bmatrix}$$

The matrix, *b*, of eigenvectors of the second row of matrices is given by

$$\begin{array}{c} \\ V \\ E \\ D \\ O \end{array} \begin{bmatrix} R & S & A \\ 0.604 & 0.604 & 0.127 \\ 0.213 & 0.213 & 0.281 \\ 0.064 & 0.064 & 0.120 \\ 0.119 & 0.119 & 0.463 \end{bmatrix}$$

The matrix, *c*, of eigenvectors of the third row of matrices is given by

$$\begin{array}{c} \\ M \\ F \\ B \end{array} \begin{bmatrix} V & E & D & O \\ 0.721 & 0.333 & 0.713 & 0.701 \\ 0.210 & 0.333 & 0.061 & 0.097 \\ 0.069 & 0.333 & 0.176 & 0.202 \end{bmatrix}$$

The final composite vector of influence on well being obtained from the product *cba* is given by

Mother: 0.635
Father: 0.209
Both: 0.156

It would seem that the therapy should depend on both the judgments and their considerable inconsistency involved. The individual was counseled to see more of his father to balance the parental influences.

2-8 ENERGY ALLOCATION (Saaty and Mariano, 1979)

In this example we are concerned with finding allocation weights for several large users of energy according to their overall contribution to different objectives in society. Let us assume the following conditions.

There are three large users of energy in the U.S.A.: C_1 = household users, C_2 = transportation, and C_3 = power generating plants. These comprise the third of lower level of the hierarchy. The objectives against which these energy users will be evaluated are: contribution to economic growth, contribution to environmental quality, and contribution to national security, which comprise the second level of the hierarchy. We construct the pairwise comparison matrix of these three objectives according to their impact on the overall objective of social and political advantage.

We have forced consistency in this case –indicating a degree of certainty in the judgments. Thus after filling in the first row, the remaining entries were derived from it, as required by the definition of consistency.

Social and political advantage

		Economic growth	Environment	National security
	Economic growth	1	5	3
M =	Environmental impact	1/5	1	3/5
	National security	1/3	5/3	1
λ_{max}	= 3.0	C.I. = 0.0	C.R. = 0.0	

When the economy is compared with the environment and then with national security, according to their socio-political impact, the economy is judged to be of strong importance in the first case and of weak importance (but still more important) in the second; hence, the values 5 and 3 in the first row, respectively. The reason for a lower number when compared with national security was thought to be due to evidence that economically poor nations are known to indulge heavily in buying weapons, but of course cannot do so without building up some financial base. The numbers in the second and third rows are obtained by requiring consistency in this case. This means, for example, that in the a_{23} position, we have economy strongly lowered over environment with value 5 and weakly favored over national security with value 3. Hence, the social-political impact of the environment over national security is 3/5 and so on. In the remaining matrices of this example we do not require consistency. The priority vector derived from this matrix is given by the column vector (which we write as a row to save space): w = (0.65, 0.13, 0.22). Thus, according to comparison of their socio-political impacts, the economy has the approximate value 0.65, the environment 0.13, and national security 0.22. Since as usual, the priority of the first hierarchy level (the overall socio-political objective) is 1, the weighted values of these priorities are equal to one times the above vector, which yields the vector itself.

Now the decision-maker, after a thorough study, has also made the following assessment of the relative importance of each user from the standpoint of the economy, the environment, and national security (the second hierarchy level). The matrices giving these judgments are, respectively

Econ.		C_1	C_2	C_3
Consumers	C_1	1	3	5
Transport	C_2	1/3	1	2
Power	C_3	1/5	1/2	1

$\lambda_{\max} = 3.00$
C.I. = 0
C.R. = 0

Env.		C_1	C_2	C_3
Consumers	C_1	1	2	7
Transport	C_2	1/2	1	5
Power	C_3	1/7	1/5	1

$\lambda_{\max} = 3.01$
C.I. = 0.01
C.R. = 0.02

N. sec.		C_1	C_2	C_3
Consumers	C_1	1	2	3
Transport	C_2	1/2	1	2
Power	C_3	1/3	1/2	1

$\lambda_{\max} = 3.01$
C.I. = 0.01
C.R. = 0.02

As above, a priority vector is derived from each matrix. They are, respectively, the three columns of the following matrix:

$$\begin{bmatrix} 0.65 & 0.59 & 0.54 \\ 0.23 & 0.33 & 0.30 \\ 0.12 & 0.08 & 0.16 \end{bmatrix}$$

This matrix is multiplied on the right by the vector *w* to weight the priority vector measuring each impact with the priority of the corresponding objective. This yields the following composite priority vector of the hierarchy level of the activities C_1, C_2, and C_3, which we seek

$$\begin{bmatrix} 0.62 \\ 0.26 \\ 0.12 \end{bmatrix}$$

Thus the overall priority of activity C_1 is 0.62, that of C_2 is 0.26, and C_3 is 0.12. We have now ranked the activities on a ratio scale according to their overall impact. This answer may appear simple, but we have to show how we get it and justify its meaningfulness.

Remark Sometimes when the weights are known measurement such as tons of pollutants or the cost of cars, one is inclined to normalize and use them instead of constructing a judgment matrix and computing the eigenvector. This process can lead to error, particularly when the utility of relative measurements to the judge are not reflected in terms of theirs ratios. For example, to a rich man, one dollar or two dollars may be about the same, yet their ratio shows greater significance.